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Construction and resolutions of certain projectively normal curves[1]

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Abstract

In this paper the (smooth irreducible) curves which can be constructed on cones over projectively normal curves $C \subseteq \mathbb{P}^r$ are studied. It is shown that all smooth curves on such cones are projectively normal and a resolution for their homogeneous ideal is given, depending on the resolution of the ideal of C.

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0. Introduction

The aim of this paper is to give a construction for certain projectively normal (p.n.) curves in $\mathbb{P}^n$, $n \geq 3$, which will also furnish a resolution of the homogeneous ideal of the curve.

The idea is to start from a projectively normal curve $C \subseteq \mathbb{P}^r$, $r \geq 2$ and to construct, on a cone $\Lambda \subseteq \mathbb{P}^{r+1}$ over C, a (smooth, irreducible) curve C_m which passes through the vertex of Λ and meets its lines m times, not counting the vertex (hence C_m is an $m:1$ covering of C).

This kind of construction is due to Jaffe (see [5]) and it is a generalization of the classical "Cayley monoidal construction" (e.g., see [1, 3, 4]).

The aim of this work is, once a bound for m in order for C_m to exist is given, to prove that C_m is p.n. (see Proposition 1) and eventually to get a resolution (which will

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not always be minimal) of the ideal sheaf of C_m from the resolution of the ideal sheaf $\mathscr{I}$ of C (see Proposition 2).

In the last section examples of this kind of construction are given.

1. Preliminaries

Let C be a smooth, irreducible, non-degenerate projectively normal curve in $\mathbb{P}^r = \mathbb{P}^r_k$, where k is an algebraically closed field with char $k = 0$. Let $\deg C = d$ and $g(C) = g$. It is quite easy to compute the degree and genus of the curve C_m described above (when it exists). On the blow-up X of the cone Λ at its vertex, we have that (by abuse of notation we will denote with C and C_m also their strict transforms on X):

$$C \equiv C_0 + dF, \qquad C_m \equiv mC_0 + (md+1)F,$$

where C_0 is the exceptional divisor and F a fiber on X, while "$\equiv$" denotes numerical equivalence.

Thus, we have

$$\deg C_m = C_m \cdot C = mC_0^2 + (md+1) + md = md + 1,$$

while $2g(C_m) - 2 = C_m^2 + C_m \cdot K_X = dm^2 + 2m + C_m \cdot (-2C_0 + (2g-2-d)F) = dm^2 + 2m - 2 + m(2g-2-d) = 2mg + 2d\binom{m}{2} - 2$, and so

$$g(C_m) = mg + d\binom{m}{2}.$$

Let us recall the necessary "ingredients" which allow the Cayley–Jaffe construction of the curve C_m to work in our case (see [5, Section 2]):

$(*)$ We need a triple (P, H, E), where P is a point on C, H a reduced hyperplane section of C with $P \in H$; $E \in |mH + P|$ on C is made of distinct points and does not meet H.

The curve C_m will appear as the section cut on Λ by a hypersurface G of degree $m+1$ such that (as schemes): $\Lambda \cap G = C_m \cup L_1 \cup \cdots \cup L_{d-1}$, where the L_i's are the lines on Λ over the points of $H - P$.

Now we want to determine a bound on m which guarantees the existence of a smooth curve C_m. Since mH is very ample on C, the only problem in $(*)$ could be that P is fixed for $|mH + P|$. Observe that we can always find E, H, P as required when mH is non-special; in fact in this case the addition of a generic point P to mH will give again a non-special divisor, so (by R.R. Theorem), $h^0(C, \mathcal{O}_C(mH + P)) = h^0(C, \mathcal{O}_C(mH)) + 1$, and P is not fixed for $|mH + P|$.

From the exact sequence

$$0 \to \mathscr{I}(m) \to \mathcal{O}_{\mathbb{P}^r}(m) \to \mathcal{O}_C(m) \to 0$$

we get that $h^1(C, \mathcal{O}_C(mH)) = h^2(\mathbb{P}^r, \mathscr{I}(m))$.

If a resolution of $\mathscr{I}$ is

$$0 \to \mathscr{F}_{r-1} \to \mathscr{F}_{r-2} \to \cdots \to \mathscr{F}_1 \to \mathscr{I} \to 0,$$

where $\mathscr{F}_i = \bigoplus_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}^r}(-\alpha_{i,j})$, $i = 1, \ldots, r-1$, then (twisting by $\mathscr{O}_{\mathbb{P}^r}(m)$ and considering the short exact sequences into which the resolution above decomposes) it is easy to see that we have $h^2(\mathbb{P}^r, \mathscr{I}(m)) = 0$ whenever $h^r(\mathbb{P}^r, \mathscr{F}_{r-1}(m)) = 0$, which we have for $m \geq \max\{\alpha_{r-1,j}\} - r$.

One could also consider the more direct fact that $h^1(C, \mathscr{O}_C(mH)) = 0$ for $md \geq 2g - 1$; we prefer to give a bound on m with respect to the $\alpha_{i,j}$'s since we are concerned with studying the graded Betti numbers (the two bounds differ only in few cases).

Better bounds for m, when a specific curve C is given, can be found with ad hoc considerations (e.g., see last section).

It can also be noticed that we can view $\mathbb{P}^r \subseteq \mathbb{P}^{r+1}$ as the hyperplane $\{w_{r+1} = 0\}$ and $H = \{l = 0\}$, so that C_m is given on Λ by a hypersurface of equation $lw_{r+1}^m - g = 0$ (which cuts $C_m \cup L_1 \cup \cdots \cup L_{d-1}$, the L_i's being the lines over the points of $H - P$ (e.g. see [5]). In this case the ideal sheaf $\mathscr{I}_{C_m}$ is (e.g. see [4])

$$(\mathscr{I}_\Lambda, lw_{r+1}^m - g) : (g).$$

This implies that when we apply this construction in the case $r = 2$, the curve C_m turns out to be directly linked to a complete intersection (Λ in a surface in $\mathbb{P}^3$), hence it is p.n. and (by the Hilbert–Burch Theorem) determinantal: its ideal sheaf $\mathscr{I}_{C_m}$ is generated by global sections which come from the maximal minors of a (2×3)-matrix of forms, whose degrees must be

$$\begin{pmatrix} d-1 & d-1 & m \\ 1 & 1 & m-d+2 \end{pmatrix}.$$

A resolution of $\mathscr{I}_{C_m}$ is given by the Eagon–Northcott complex:

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-m-d) \oplus \mathscr{O}_{\mathbb{P}^3}(-m-2) \to \mathscr{O}_{\mathbb{P}^3}(-d) \oplus \left(\bigoplus^{2} \mathscr{O}_{\mathbb{P}^3}(-m-1) \right) \to \mathscr{I}_{C_m} \to 0.$$

Notice that the resolution above (hence the ideal generation) could be non-minimal (when C_m is a complete intersection, e.g. for $d = 3$, $m = 1$).

The considerations in the next section can be viewed as a generalization of what we have just seen for $r = 2$ to the case $r \geq 3$, where we no longer have liaison to get that C_m is p.n. nor the Hilbert–Burch theorem to find a resolution.

Let us notice that the ideal of such C_m do not arise, in general, by taking the Hilbert–Burch matrix of the base curve C in $\mathbb{P}^3$, viewing it over the coordinate ring of $\mathbb{P}^4$, and adding a suitable column (e.g., see Section 3).

2. Projective normality and a resolution of C_m

With notation as in Section 1, let us consider curves C_m, $m>0$, on X, with $C_m \in |mC_0+(md+1)F|=|mC+F|$. Let $V=H^0(X,\mathcal{O}_X(C))$, and $S=S(V)=k[x_0,x_1,x_2,\dots,x_{r+1}]$, so we view $\mathbb{P}^{r+1}$ as **Proj**(S).

Let $\boldsymbol{B}=\bigoplus_q H^0(X,\mathcal{O}_X(qC))$; then notice that a resolution of $\boldsymbol{B}$ as an S-module is known (it corresponds to a resolution of $\bigoplus_q H^0(X,\mathcal{O}_A(qC)))$, and it has the same graded Betti numbers as those of $C\subseteq\mathbb{P}^r$. Hence, we have

$$0\to F_{r-1}\to F_{r-2}\to\cdots\to F_1\to S\to \boldsymbol{B}\to 0,$$

where $F_i=\bigoplus_{j=1}^{\beta_i} S(-\alpha_{i,j})$, $i=1,\dots,r-1$.

We want to show, when C_m is irreducible and smooth, that its image in $\mathbb{P}^{r+1}$ (which is isomorphic to C_m, since $C_m\cdot C_0=1$, and which we will denote by C_m again) is a projectively normal curve. Moreover, we look for a resolution of $\boldsymbol{A}_m=\bigoplus_q H^0(X,\mathcal{O}_{C_m}(qC))$ as an S-module (if C_m is p.n. $\boldsymbol{A}_m$ is its ring of coordinates).

Let us make several remarks.

Remark 1. Let $\boldsymbol{A}_F=\bigoplus_q H^0(X,\mathcal{O}_F(qC))=\bigoplus_q H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(q))$; a resolution of $\boldsymbol{A}_F$ as an S-module is well known (by viewing F as a line in $\mathbb{P}^{r+1}$):

$$0\to L_r\to L_{r-1}\to\cdots\to L_1\to\mathcal{I}\to S\to \boldsymbol{A}_F\to 0,$$

where $L_i=\bigoplus^{\binom{r}{i}} S(-i)$, $i=1,\dots,r$.

Remark 2. From the exact sequence

$$0\to\mathcal{O}_X(qC-F)\to\mathcal{O}_X(qC)\to\mathcal{O}_F(qC)\to 0$$

since, $\forall q\ge -1$, we have $H^1(F,\mathcal{O}_F(qC))=H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(q))=0$, and since $\mathcal{O}_X(qC)\to\mathcal{O}_F(qC)$ is surjective at the H^0-level, it follows that

$$H^1(X,\mathcal{O}_X(qC-F))=H^1(X,\mathcal{O}_X(qC)),\quad \forall q\ge -1.$$

Remark 3. Let $q\ge 0$; from the exact sequence

$$0\to\mathcal{O}_X((q-1)C)\to\mathcal{O}_X(qC)\to\mathcal{O}_C(qC)\to 0,$$

since $C\subseteq\mathbb{P}^r$ is p.n., and $h^2(X,\mathcal{O}_X((q-1)C))=h^0(X,\mathcal{O}_X((1-q)C+K_X))=0$, we get exactness at H^1-level, i.e.

$$h^1(X,\mathcal{O}_X(qC))=h^1(X,\mathcal{O}_X((q-1)C))+h^1(C,\mathcal{O}_C(qC)).$$

Remark 4. $\forall q\le -1$, $H^1(X,\mathcal{O}_X(qC-F))=0$.

This comes from the Kodaira Vanishing Theorem (note that $-qC+F$ is ample).

Remark 5. A generic curve $\Gamma_m \in |mC|$ on X is smooth and irreducible, and its image in $\mathbb{P}^{r+1}$ (which we still will denote by Γ_m) is cut out on X by a hypersurface of degree m, so it is projectively normal.

Proposition 1. *Let C, Λ be as in Section 1. Then all the smooth, irreducible curves on Λ are projectively normal (in particular, $\forall m \geq \max\{\alpha_{i,j}\} - r$ there is a smooth irreducible curve C_m on Λ which is p.n).*

Proof. It is easy to check that the only smooth irreducible curves on Λ are those of type C_m, $m \geq 0$ or Γ_m, $m \geq 1$, which come from divisors of type $mC_0 + (md+1)F$, or $mC_0 + mdF$ on X, respectively (those have intersection 0 or 1 with C_0; all the others will be singular at the vertex).

We already noticed that the curves Γ_m are p.n. (Remark 5), so let us consider the curves C_m; it is enough to show that one curve in $|C_m|$ on Λ is p.n. to have that all the smooth curves in the class are p.n. In fact, C_m is p.n. if and only if $h^1(\mathbb{P}^{r+1}, \mathscr{I}_{C_m}(q)) = 0$, $\forall q \in \mathbb{Z}$; let us check that this fact depends only on the class $|C_m|$. Consider the exact sequence

$$0 \to \mathscr{I}_\Lambda(q) \to \mathscr{I}_{C_m}(q) \to \mathscr{I}_{C_m,\Lambda}(q) \to 0,$$

where $\mathscr{I}_{C_m,\Lambda} = \mathcal{O}_\Lambda(-C_m)$ is the ideal sheaf of C_m in $\mathcal{O}_\Lambda$.

Since Λ is p.C.M., we have that $h^1(\mathbb{P}^{r+1}, \mathscr{I}_\Lambda(q)) = h^2(\mathbb{P}^{r+1}, \mathscr{I}_\Lambda(q)) = 0$, $\forall q \in \mathbb{Z}$, and so we get that $h^1(\mathbb{P}^{r+1}, \mathscr{I}_{C_m}(q)) = h^1(\mathbb{P}^{r+1}, \mathscr{I}_{\Lambda,C_m}(q))$, where the second depends only on the class of C_m as a divisor on Λ.

Now let us show that C_m is p.n.; in the linear system $|C_m| = |\Gamma_m + F|$ we can consider ($\forall m \geq 1$) a reduced and reducible curve $C_m = \Gamma_m \cup F$ given by a smooth, irreducible Γ_m and a line F, with $\Gamma_m \cap F$ given by m distinct points.

Since $\forall q \geq m$ the sequence

$$0 \to \mathcal{O}_{\Gamma_m \cup F}(qC) \to \mathcal{O}_{\Gamma_m}(qC) \oplus \mathcal{O}_F(qC) \to \mathcal{O}_{\Gamma_m \cap F}(qC) \to 0$$

is exact both at the H^0 and at the H^1 level, then, for $q \geq m$, we get

$$h^0(X, \mathcal{O}_{C_m}(qC)) = h^0(\Gamma_m, \mathcal{O}_{\Gamma_m}(qC)) + q - m + 1, \tag{1}$$

$$h^1(X, \mathcal{O}_{C_m}(qC)) = h^1(\Gamma_m, \mathcal{O}_{\Gamma_m}(qC)). \tag{2}$$

Now consider the exact sequence

$$0 \to \mathcal{O}_X((q-m)C - F) \to \mathcal{O}_X(qC) \to \mathcal{O}_{C_m}(qC) \to 0. \tag{3}$$

We want to show that (3) is H^0-exact, $\forall q \in \mathbb{Z}$. If $q-m<0$, this is true by Remark 4; when $q \geq m$, consider the exact sequence

$$0 \to \mathcal{O}_X((q-m)C) \to \mathcal{O}_X(qC) \to \mathcal{O}_{\Gamma_m}(qC) \to 0. \tag{4}$$

We have that (4) is exact at the H^0 and H^1 level, so, from Remark 2, and by (1) and (2) we have that also (3) is exact at the H^0 and H^1 level.

This implies that for the embedded curve $C_m \subseteq \mathbb{P}^{r+1}$ the map

$$H^0(\mathbb{P}^{r+1}, \mathcal{O}_{\mathbb{P}^{r+1}}(q)) \to H^0(\mathbb{P}^{r+1}, \mathcal{O}_{C_m}(q))$$

is surjective, hence $H^1(\mathbb{P}^{r+1}, \mathcal{I}_{C_m}(q)) = 0$, $\forall q \in \mathbb{Z}$.

Then, when we choose an m such that one (hence the generic) curve $C_m \in |mC+F|$ is irreducible and smooth, we will have again that $H^1(\mathbb{P}^{r+1}, \mathcal{I}_{C_m}(q)) = 0$, $\forall q$, since this property is an open one in the Hilbert scheme of curves of given genus and degree of $\mathbb{P}^{r+1}$, and the reduced curve we have considered before is a deformation of the generic curve in $H^0(\mathbb{P}^{r+1}, \mathcal{O}_\Lambda(C_m))$. Thus, $C_m \subseteq \mathbb{P}^{r+1}$ is projectively C.M., hence projectively normal. □

Now, let $C_m \subseteq \Lambda \subseteq \mathbb{P}^{r+1}$ be a (smooth, irreducible, p.n.) curve as before, and consider the exact sequence

$$0 \to \mathcal{O}_X((q-m)C - F) \to \mathcal{O}_X((q-m)C) \to \mathcal{O}_F((q-m)C) \to 0.$$

Since resolutions of A_F (see Remark 1) and of $\boldsymbol{B} = \bigoplus_q H^0(X, \mathcal{O}_X(qC))$ are known, and the above sequence is exact at the H^0 level (see Remark 2), in order to find a resolution of $\boldsymbol{C}_m = \bigoplus_q H^0(X, \mathcal{O}_X((q-m)C - F))$ as S-module, we can consider the short exact sequences which fit into the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
0 \to \boldsymbol{C}_m & \to & \boldsymbol{B}(-m) & \to & A_F(-m) & \to & 0 \\
 & & \uparrow & & \uparrow & & \\
 & 0 \to & S(-m) & \to & S(-m) & \to & 0 \\
 & & \uparrow & & \uparrow & & \\
 & & N & \longrightarrow & K & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

The resolution for N (resp. K) comes from the rest of the resolution for $\boldsymbol{B}(-m)$ (resp. $A_F(-m)$). We thus have (from the Snake Lemma) a short exact sequence

$$0 \to N \to K \to \boldsymbol{C}_m \to 0$$

and we know the resolution of N and of K. The mapping cone construction then gives a resolution for C_m:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_m & \longrightarrow & B(-m) & \longrightarrow & A_F(-m) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & L_1(-m) & & S(-m) & & S(-m) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & L_2(-m)\oplus F_1(-m) & & F_1(-m) & & L_1(-m) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & L_3(-m)\oplus F_2(-m) & & F_2(-m) & & L_2(-m) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & L_r(-m)\oplus F_{r-1}(-m) & & F_{r-1}(-m) & & L_{r-1}(-m) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & L_r(-m) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
$$

(We thank the referee for suggesting this proof, which is more direct than the one we had originally given.)

Then consider the exact sequence

$$0 \to \mathcal{O}_X((q-m)C - F) \to \mathcal{O}_X(qC) \to \mathcal{O}_{C_m}(qC) \to 0$$

(which is exact at the H^0-level since C_m is p.n.) and again use a mapping cone in order to compute a resolution (may be not minimal) of A_m:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_m & \longrightarrow & B & \longrightarrow & A_m & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & L_1(-m) & & S & & S & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & L_2(-m)\oplus F_1(-m) & & F_1 & & L_1(-m)\oplus F_1 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & L_3(-m)\oplus F_2(-m) & & F_2 & & L_2(-m)\oplus F_1(-m)\oplus F_2 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & L_r(-m)\oplus F_{r-1}(-m) & & F_{r-1} & & L_{r-1}(-m)\oplus F_{r-2}(-m)\oplus F_{r-1} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & L_r(-m)\oplus F_{r-1}(-m) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
$$

So one gets (via sheafifying the resolution of A_m):

Proposition 2. *Let C_m be as above, then a resolution (maybe non-minimal) of the ideal sheaf $\mathcal{I}_{C_m} \subseteq \mathcal{O}_{\mathbb{P}^{r+1}}$ is*

$$0 \to \mathcal{O}_{\mathbb{P}^{r+1}}(-m-r) \oplus \left(\bigoplus_{j=1}^{\beta_{r-1}} \mathcal{O}_{\mathbb{P}^{r+1}}(-m-\alpha_{r-1,j}) \right)$$

$$\to \left(\bigoplus^{r} \mathcal{O}_{\mathbb{P}^{r+1}}(-m-r+1) \right) \oplus \left(\bigoplus_{j=1}^{\beta_{r-1}} \mathcal{O}_{\mathbb{P}^{r+1}}(-\alpha_{r-1,j}) \right) \oplus \left(\bigoplus_{j=1}^{\beta_{r-2}} \mathcal{O}_{\mathbb{P}^{r+1}}(-m-\alpha_{r-2,j}) \right)$$

$$\cdots$$

$$\to \left(\bigoplus^{\binom{r}{2}} \mathcal{O}_{\mathbb{P}^{r+1}}(-m-2) \right) \oplus \left(\bigoplus_{j=1}^{\beta_2} \mathcal{O}_{\mathbb{P}^{r+1}}(-\alpha_{2,j}) \right) \oplus \left(\bigoplus_{i=1}^{\beta_1} \mathcal{O}_{\mathbb{P}^{r+1}}(-m-\alpha_{1,j}) \right)$$

$$\to \left(\bigoplus^{r} \mathcal{O}_{\mathbb{P}^{r+1}}(-m-1) \right) \oplus \left(\bigoplus_{i=1}^{\beta_1} \mathcal{O}_{\mathbb{P}^{r+1}}(-\alpha_{1,j}) \right) \to \mathcal{I}_{C_m} \to 0.$$

3. Examples

3.1. Curves in $\mathbb{P}^4$

May be the most interesting case of our construction is for $r=3$, since the resolutions of p.n. curves in $\mathbb{P}^3$ is well known, while they are not so easy to compute for p.n. curves in $\mathbb{P}^4$.

By the Hilbert–Burch Theorem, every projectively normal curve $C \subseteq \mathbb{P}^3$ is determinantal, and so its ideal sheaf $\mathcal{I}$ is generated by global sections which come from the maximal minors of a $\rho \times (\rho+1)$-matrix of forms, and a resolution of $\mathcal{I}$ is given by the Eagon–Northcott complex

$$0 \to \bigoplus_{j=1}^{\rho} \mathcal{O}_{\mathbb{P}^3}(-n_j) \to \bigoplus_{i=1}^{\rho+1} \mathcal{O}_{\mathbb{P}^3}(-d_i) \to \mathcal{I} \to 0.$$

When $C_m \subseteq \mathbb{P}^4$ exists, we get a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-m-3) \oplus \left(\bigoplus_{j=1}^{\rho} \mathcal{O}_{\mathbb{P}^4}(-m-n_j) \right)$$

$$\to \left(\bigoplus^{3} \mathcal{O}_{\mathbb{P}^4}(-m-2) \right) \oplus \left(\bigoplus_{i=1}^{\rho+1} \mathcal{O}_{\mathbb{P}^4}(-m-d_i) \right) \oplus \left(\bigoplus_{j=1}^{\rho} \mathcal{O}_{\mathbb{P}^4}(-n_j) \right)$$

$$\to \left(\bigoplus^{3} \mathcal{O}_{\mathbb{P}^4}(-m-1) \right) \oplus \left(\bigoplus_{i=1}^{\rho+1} \mathcal{O}_{\mathbb{P}^4}(-d_i) \right) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{C_m} \to 0.$$

In general, C_m can be constructed when $m \geq \max_j\{n_j\} - 3$, but there are cases when the construction can be carried on also for smaller values of m (for which mH is special); for example, let C be a C_5^7; in this case $|H|$ has degree $7 = 2g - 3$, and index of speciality $= 1$, hence its hyperplane divisors can be viewed as obtained from a canonical one minus one point, so if one adds that point, say P, to a divisor H, the system $|H + P| = |K_C|$ has again index of speciality $= 1$, i.e. P is not fixed for it and condition $(*)$ holds, hence we can construct a curve $C_1 = C_5^8 \subseteq \mathbb{P}^4$ on a cone over C.

Notice that C_1 is a canonical curve and is non-trigonal (otherwise it would have ∞^1 trisecants and C should have a node, being its projection from one point), so it is generated by quadrics (it is actually the complete intersection of three quadrics) and it is immediate to check that the resolution shown above is not minimal for it.

Hence there are cases when the resolution we gave is actually non-minimal; anyway we notice that it will surely be minimal (for trivial reasons) when $m \geq \max_j\{n_j\}$, and $d_i \neq 3, n_j, \ \forall i,j$.

3.2. *Curves over complete intersections in* $\mathbb{P}^3$

Let $C \subseteq \mathbb{P}^3$ be the complete intersection of two surfaces of degree a,b; then the resolution of C is

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a-b) \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-b) \to \mathcal{I} \to 0,$$

hence mH is non-special for $m \geq a + b - 3$ and for these values of m we can obtain a curve C_m on Λ.

A resolution of C_m is

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-a-b-m) \oplus \mathcal{O}_{\mathbb{P}^4}(-m-3)$$

$$\to \mathcal{O}_{\mathbb{P}^4}(-b-m) \oplus \mathcal{O}_{\mathbb{P}^4}(-a-m) \oplus \mathcal{O}_{\mathbb{P}^4}(-a-b) \oplus \left(\bigoplus^{3} \mathcal{O}_{\mathbb{P}^4}(-m-2) \right)$$

$$\to \mathcal{O}_{\mathbb{P}^4}(-a) \oplus \mathcal{O}_{\mathbb{P}^4}(-b) \oplus \left(\bigoplus^{3} \mathcal{O}_{\mathbb{P}^4}(-m-1) \right) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{C_m} \to 0.$$

It is easy to check that this resolution has to be minimal when $a,b \geq 4$ and $m \geq a + b$, but those bounds are far from being sharp, as the following examples will show.

Let us consider the case $a=b=2$: when $m=1$ (hence C and C_1 are elliptic normal curves), the resolution above is not minimal, since the minimal one is

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-5) \to \bigoplus^5 \mathcal{O}_{\mathbb{P}^4}(-3) \to \bigoplus^5 \mathcal{O}_{\mathbb{P}^4}(-2) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{C_1} \to 0.$$

For each $m \geq 2$, instead, the resolution is minimal; this is obvious for $m=2$ and $m \geq 4$, while for $m=3$ we have that $C_3 = C_{15}^{13} \subseteq \mathbb{P}^4$ is a curve of degree 13 and genus 15 for which we get a resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-7) \oplus \mathcal{O}_{\mathbb{P}^4}(-6) \to \left(\bigoplus^5 \mathcal{O}_{\mathbb{P}^4}(-5)\right) \oplus \mathcal{O}_{\mathbb{P}^4}(-4)$$

$$\to \left(\bigoplus^3 \mathcal{O}_{\mathbb{P}^4}(-4)\right) \oplus \left(\bigoplus^2 \mathcal{O}_{\mathbb{P}^4}(-2)\right) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{C_3} \to 0.$$

Is this resolution minimal? One could think that maybe it is not; that there is a redundant addendum $\mathcal{O}_{\mathbb{P}^4}(-4)$ (i.e. a redundant syzygy and a redundant generator in degree 4). So the question is: can two forms of degree 4 (and two of degree 2) be enough in order to generate the homogeneous ideal of C_3? The answer, which implies that the above resolution is minimal, is “No”. One can check this in the following way: we can work on the ideal of a generic hyperplane section Z of C_3 (which will have the same graded Betti numbers since C_3 is p.n); Z is reduced and consists of 13 points in $\mathbb{P}^3$ which are contained in an elliptic curve E_4, hence on two quadric surfaces.

Consider a smooth quadric Q containing Z and a plane model of it: then the hyperplane sections of Q are given on the plane by the conics through two points P_1, P_2, while Z corresponds to 13 other points $P_3, \dots, P_{15}$.

The other quadric containing Z gives on the plane a quartic curve D passing with multiplicity 2 through P_1, P_2 and simply through $P_3, \dots, P_{15}$; the quartic surfaces through Z correspond to the linear system of curves of degree eight containing $P_3, \dots, P_{15}$ and having P_1, P_2 as (at least) quadruple points. A simple computation shows that this linear system contains at least $45 - 33 = 12$ independent curves; among them at most nine are of type $D \cup D'$, where D' is another quartic containing P_1, P_2 as double points, hence there must be at least three independent curves not composed with D; those correspond to the three quartic surfaces through Z in $\mathbb{P}^3$ which are needed as generators.

Hence the resolution is minimal.

3.3. *Bielliptic curves*

A *bielliptic curve* is a (smooth, irreducible, non-hyperelliptic) curve $\mathscr{C}$ endowed with a 2 : 1 morphism $\phi : \mathscr{C} \to E$, where E is a (smooth, irreducible) elliptic curve.

Let us consider an elliptic normal curve $E_{r+1} \in \mathbb{P}^r$; for $m=2$, we get that we can obtain two bielliptic curves Γ_2 and C_2 in $\mathbb{P}^{r+1}$ on a cone over E_{r+1}. Γ_2 is actually a canonical bielliptic curve C^{2r+2}_{r+2}, and its resolution is known [6] (see also [2]).

Also C_2 is bielliptic and p.n. It has degree $2r+3$ and genus $r+3$ and it can be viewed as the projection of a canonical bielliptic curve $C^{2r+4}_{r+3} \in \mathbb{P}^{r+2}$ from one of its points.

Since the resolution of the ideal of E_{r+1} is known (it has the same Betti numbers as $r+1$ generic points in $\mathbb{P}^r$), one can compute a resolution for these curves using Proposition 2.

For $r=3$, hence $C_2 = C^9_6 \subseteq \mathbb{P}^4$, we get the minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-6) \oplus \mathcal{O}_{\mathbb{P}^4}(-5) \to \bigoplus^{5} \mathcal{O}_{\mathbb{P}^3}(-4)$$
$$\to \left(\bigoplus^{3} \mathcal{O}_{\mathbb{P}^3}(-3) \right) \oplus \left(\bigoplus^{2} \mathcal{O}_{\mathbb{P}^4}(-2) \right) \to \mathcal{I}_{C_2} \to 0.$$

For $r=4$, so $C_2 = C^{11}_7 \subseteq \mathbb{P}^4$, we get

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-7) \oplus \mathcal{O}_{\mathbb{P}^4}(-6) \to \bigoplus^{10} \mathcal{O}_{\mathbb{P}^3}(-5)$$
$$\to \left(\bigoplus^{11} \mathcal{O}_{\mathbb{P}^3}(-4) \right) \oplus \left(\bigoplus^{5} \mathcal{O}_{\mathbb{P}^4}(-3) \right)$$
$$\to \left(\bigoplus^{4} \mathcal{O}_{\mathbb{P}^3}(-3) \right) \oplus \left(\bigoplus^{5} \mathcal{O}_{\mathbb{P}^4}(-2) \right) \to \mathcal{I}_{C_2} \to 0.$$

In this case the resolution could be non-minimal; the question is whether we need the four cubics among the generators.

The five quadrics intersect in the elliptic normal cone Λ on E_5, hence a cubic surface will cut on it a curve $D = C_2 \cup L_1 \cup \cdots \cup L_4$, where the L_i's and the tangent line to C_2 at V, the vertex of the cone, cuts on E_5 a hyperplane divisor (this is the Cayley–Jaffe construction; see [5]).

Hence, the curve D has, at V, a tangent space of dimension 4 and the intersection with any other cubic surface through C_2 can lower this dimension only by one, so we need at least another three cubic surfaces to get the tangent line of C_2.

Thus, we need four cubic forms to generate the ideal of C_2, and the resolution is minimal.

References

[1] A. Cayley, Considerationés générales sur les courbes en espace, Compte Rendu 54 (1862) 55–60, 396–400, 672–678.

[2] A. Del Centina, A. Gimigliano, Scrollar invariants and resolutions of certain d-gonal curves, Ann. Univ. Ferrara Sez. VII - Sc. Mat. XXXIX (1993) 187–201.

[3] F. Gaeta, Nuove ricerche sulle curve sghembe di residuale finito e sui gruppi di punti del piano, Ann. Math. Pura Appl. Ser. IV XXXI (1950) 1–64.

[4] A. Gimigliano, Varietá algebriche differenze di intersezioni complete insiemistiche e ideali divisione loro associati, Alg. Geom. Suppl. BUMI 2 (1980) 259–269.

[5] D.B. Jaffe, Smooth curves on a cone which pass through its vertex, Manuscripta Math. 73 (1991) 187–205.

[6] F.O. Schreyer, Syzygies of canonical curves and special linear series, Math. Ann. 275 (1986) 105–137.